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On the integrability of the extended nonlinear Schrödinger equation and the coupled extended nonlinear Schrödinger equations

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Abstract. We consider the extended nonlinear Schrödinger (ENLS) equation which governs the propagation of nonlinear optical fields in a fibre with higher-order effects such as higher-order dispersion and self-steepening. We show that the ENLS equation does not pass the Painlevé test. Similarly, we claim that the coupled ENLS equations and N-coupled ENLS equations which govern the simultaneous propagation of two and more nonlinear fields in optical fibres are also not integrable from the Painlevé analysis point of view.

It is well known that the dynamics of nonlinear wave propagation in a single-mode fibre is governed by the famous nonlinear Schrödinger (NLS) equation [1–3]. For the transmission of more channels, time division multiplexing is normally used, which can be effectively achieved by the propagation of ultrashort pulses. Propagation of short pulses induces higher-order effects such as higher-order dispersion, self-steepening and stimulated inelastic scattering [2,3]. With the effects of higher-order dispersion and self-steepening, the wave propagation is governed by the extended NLS (ENLS) equation of the form [4]

$$\mathrm{i}q_z - \frac{k''}{2}q_{tt} + \beta|q|^2q - \mathrm{i}\frac{k'''}{6}q_{ttt} + \mathrm{i}\gamma(|q|^2q)_t = 0 \tag{1}$$

where q is the slowly varying envelope of the axial field, k'', β, k''' and γ are group velocity dispersion, self-phase modulation, higher-order dispersion and self-steepening parameters respectively, and subscripts z and t denote spatial and temporal partial derivatives.

Liu and Wang [4] considered the above form of ENLS equation and derived the exact one- and two-soliton solutions using the Hirota bilinear method [5] under the conditions $3k''\gamma = \beta k'''$ and $k''\gamma = \beta k'''$. In [6], coupled ENLS equations of the following form for the propagation of two fields simultaneously are considered:

$$\begin{aligned} &\mathrm{i}q_{1z} - \frac{k''}{2}q_{1tt} + \beta[(x|q_1|^2 + y|q_2|^2)q_1] - \mathrm{i}\frac{k'''}{6}q_{1ttt} + \mathrm{i}\gamma[(x|q_1|^2 + y|q_2|^2)q_1]_t = 0 \\ &\mathrm{i}q_{2z} - \frac{k''}{2}q_{2tt} + \beta[(y|q_1|^2 + x|q_2|^2)q_2] - \mathrm{i}\frac{k'''}{6}q_{2ttt} + \mathrm{i}\gamma[(y|q_1|^2 + x|q_2|^2)q_2]_t = 0 \end{aligned} \tag{2}$$

where x and y are the coupling coefficients between the self-phase modulation and the cross phase modulation. In [6], under the condition $k''\gamma = \beta k'''$, one- and two-soliton solutions for

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the coupled ENLS equations are derived using the Hirota bilinear method (for $x = y = 1$). In this paper, we show that the ENLS equation (1) does not pass the Painlevé test. Finally, we claim that the coupled ENLS equations (2) and N-coupled ENLS equations are also not integrable from the Painlevé singularity analysis point of view.

A new set of variables $a(=q)$ and $b(=q^*)$ are introduced for the purpose of Painlevé singularity structure analysis [7]. Thus, using equation (1), a and b can be written as

$$\begin{aligned} &\mathrm{i}a_z - \frac{k''}{2}a_{tt} + \beta a^2 b - \mathrm{i}\frac{k'''}{6}a_{ttt} + \mathrm{i}\gamma(a^2b)_t = 0 \\ &-\mathrm{i}b_z - \frac{k''}{2}b_{tt} + \beta b^2 a + \mathrm{i}\frac{k'''}{6}b_{ttt} - \mathrm{i}\gamma(b^2a)_t = 0. \end{aligned} \tag{3}$$

Generalized Laurent series expansions of a and b are

$$\begin{aligned} a &= \phi^{\alpha_1}\sum_{j=0}^{\infty} a_j(z,t)\phi^j \\ b &= \phi^{\alpha_2}\sum_{j=0}^{\infty} b_j(z,t)\phi^j \end{aligned} \tag{4}$$

with $a_0, b_0 \neq 0$, where α_1 and α_2 are negative integers, a_j and b_j are sets of expansion coefficients which are analytic in the neighbourhood of the noncharacteristic singular manifold. Looking at leading order, $a \approx a_0\phi^{\alpha_1}$ and $b \approx b_0\phi^{\alpha_2}$ are substituted in equation (3) and upon balancing dominant terms, the following results are obtained:

$$\alpha_1 = \alpha_2 = -1 \qquad a_0 b_0 = \frac{k'''}{3\gamma}\phi_t^2. \tag{5}$$

Substituting the full Laurent series and considering leading-order terms alone, we obtain the following equation:

$$\begin{pmatrix} A & Ba_0^2 \\ -Bb_0^2 & -A \end{pmatrix}\begin{pmatrix} a_j \\ b_j \end{pmatrix} = 0 \tag{6}$$

where

$$\begin{aligned} A &= \frac{-k'''}{6}(j-1)(j-2)(j-3)\phi_t^3 + 2\gamma(j-3)a_0b_0\phi_t \\ B &= \gamma(j-3)\phi_t. \end{aligned}$$

On solving equation (6), the resonance values are found to be

$$j = -1, 0, 3, 3, 3, 4. \tag{7}$$

The resonance value at $j = -1$ represents the arbitrariness of the singularity manifold $\phi(z, t)$, while the resonance at $j = 0$ is associated with the arbitrariness of the functions a_0 and b_0 (as seen in equation (5)). Degeneracy of the resonance value at $j = 3$, repeating three times as in equation (7), claims three arbitrary functions on the functions a_3 and b_3. It is obvious that there cannot be three arbitrary functions over two functions. From this, it is very clear that the ENLS equation (1) does not pass the Painlevé test. So, from the Painlevé analysis point of view, it is clear that the ENLS equation is not integrable. However, it has exact one- and two-soliton solutions through the Hirota bilinear method [4]. It is a known fact that the Hirota bilinear method usually generates one- and two-soliton solutions and does not give any idea about integrability. For nonintegrable equations like the ENLS equation, the Hirota bilinear method only gives trouble during the calculation of higher-soliton solutions.

Now, let us discuss the integrability of the coupled ENLS equations (2) and N-coupled ENLS equations. Coupled ENLS equations (2) and N-coupled ENLS equations can be reduced to the ENLS equation (1) for $q_1 = q_2 \ldots q_N = q$. As the ENLS equation (1) does not pass the Painlevé test, we claim that the coupled ENLS equations (2) and N-coupled ENLS equations are not integrable from the Painlevé analysis point of view. But the coupled ENLS equations (2) have exact one- and two-soliton solutions through the Hirota bilinear method [6]. We are sure that one can also generate the one- and two-soliton solutions using the Hirota bilinear method for the N-coupled ENLS equations.

In the second part of [6], the authors have considered a different kind of coupled higher-order NLS equation which includes the effect of delayed nonlinear response and derived the exact one- and two-soliton solutions, once again using the Hirota bilinear method. It is interesting to note that coupled equations pass the Painlevé test [8] and for the conformation of complete integrability, it has the linear eigenvalue problem and associated soliton solutions [9].

In this paper, we have reported that the ENLS equation and its coupled forms do not pass the Painlevé test. So, we conclude that the ENLS equation and its coupled forms are not integrable.

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